Exercise 80

The graph of any quadratic function $f(x) = ax^2 + bx + c$ is a parabola. Prove that the average of the slopes of the tangent lines to the parabola at the endpoints of any interval $[p, q]$ equals the slope of the tangent line at the midpoint of the interval.

Solution

Take the derivative of the function representing the parabola.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(ax^2 + bx + c) \\ &= \frac{d}{dx}(ax^2) + \frac{d}{dx}(bx) + \frac{d}{dx}(c) \\ &= a\frac{d}{dx}(x^2) + b\frac{d}{dx}(x) + \frac{d}{dx}(c) \\ &= a(2x) + b(1) + (0) \\ &= 2ax + b \end{aligned}$$

The slope at the endpoint $x = p$ is

$$f'(p) = 2ap + b,$$

the slope at the endpoint $x = q$ is

$$f'(q) = 2aq + b,$$

and the slope at the midpoint $x = (p + q)/2$ is

$$f'\left(\frac{p+q}{2}\right) = 2a\left(\frac{p+q}{2}\right) + b = a(p + q) + b.$$

Take the average of slopes at the endpoints.

$$\frac{f'(p) + f'(q)}{2} = \frac{(2ap + b) + (2aq + b)}{2} = \frac{2a(p + q) + 2b}{2} = a(p + q) + b$$

Therefore, the average of the slopes of the tangent lines to the parabola at the endpoints of any interval $[p, q]$ equals the slope of the tangent line at the midpoint of the interval.

$$\frac{f'(p) + f'(q)}{2} = f'\left(\frac{p+q}{2}\right)$$